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Critical dynamics and universality in kinetic Ising models without translational invariance

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Abstract. The critical dynamics of the Glauber–Ising model on non-translationally invariant lattices is studied. Both a quasi-periodic and a fractal geometry are considered. The distribution of inverse relaxation times $\rho(1/\tau)$ is calculated using a generating function method. The distribution consists of bands with an internal self-similar structure. In the limit $1/\tau \to 0$, $\rho(1/\tau)$ diverges with a universal exponent related to the dynamic critical exponent z. The width of the lowest frequency band is determined by a non-universal bare time scale, which is related to the presence of metastable states.

1. Introduction

Second-order phase transitions are characterized by long-range fluctuations and these fluctuations are responsible for critical singularities. The nature of the interactions between the long-range fluctuations depends only on a small number of general properties of the Hamiltonian. Therefore, different systems that share the same symmetry of the Hamiltonian in spin-space and in coordinate-space are expected to have thermodynamic quantities with similar singular behaviour [1]. This idea of 'universality' has also been extended to the description of time-dependent phenomena near the critical point [2]. The universality classes that describe the critical dynamics should be a subdivision of those describing the statics according to conservation laws that restrict the dynamics [3].

It is important to note that the term 'universal' is applied only to some of the characteristics of a system. For example, we expect the relaxation towards equilibrium to be characterized by a time scale $\tau_c = A\xi^z(T)$, where $\xi(T)$ is the static correlation length that diverges at the critical temperature, T_c. In this relation z is a universal dynamic exponent, but A is a non-universal amplitude which depends on the fine details of the system. In general, the value of z can be obtained from the fixed point of a scale transformation on the system but it is usually very difficult to find the exact expression [4] for the amplitude A. Thus, in practice, a phenomenological description of the dynamics is often used, where A is assumed to be a smooth function of temperature and hence a constant at T_c. The amplitude A is usually taken to be proportional to some 'bare time scale', τ_0.

The assumption that τ_0 should be considered as a constant is often a reasonable one. However, there are models where this assumption is not valid. These models are characterized by the presence of metastable states in the dynamics which, although they have zero weight in the static partition function, have very long relaxation times. Therefore, even

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in a phenomenological theory, these metastable states may cause a strongly T-dependent or singular τ_0. In models with metastable states one finds two distinct time scales: one is linked to the cooperative behaviour characterizing the critical point of the system and is universal while the other is a result of short-range details due to a particular transition mechanism/rate.

A simple example of a model having metastable states is the Glauber–Ising chain with alternating bond strengths [5, 6] where the relaxation is constrained to occur via single-spin flips [7]. The metastable states are the configurations that consist of pairs of neighbouring spins coupled by the strong bond and pointing in the same direction. The flipping of a pair can be achieved via single-spin flips only through the intermediate state where these two spins are pointing in opposite directions. The time scale for this transition, as given by the corresponding Arrhenius law [8], diverges as the critical point $T_c = 0$ is approached. Therefore, the critical slowing down and divergence of τ_0 occur simultaneously. However, these are two different mechanisms and should be separated. The short-range time scale has nothing to do with critical phenomena. Any small system with only a few spins will have these non-relaxing metastable states, and the apparent non-critical 'slowing down'. Only small wavenumber, k, components in a very large system will show universal critical relaxation. It is important to note that in many models the existence of the metastable states is a direct result of the single-spin-flip restriction. In a previous communication [9] we have shown that the addition of the multi-spin flips allows the metastable states to relax, and the divergent short-range time scale associated with the single-spin-flip dynamics disappears.

The different roles the two time scales play are clearly evident on lattices which are translationally invariant. In this case the dynamics can be studied using Laplace and Fourier transform methods. For example, the dispersion relation $\tau^{-1}(k)$ for the Laplace component of the average magnetization [6] of an Ising chain with a unit cell composed of three spins interacting with bonds of strength $\{K_A, K_A, K_B\}$, $K_A > K_B$ is shown in figure 1(a). The periodicity of the unit cell leads to three bands, separated by gaps. In the limit of low T the width of the bands is proportional to $\tau_0^{-1} = e^{-2(K_A - K_B)}$. Each mode in the lowest band corresponds to a state which is metastable with respect to single-spin flips and has a relaxation time that diverges at $T = 0$. All of these states have the three spins located between the weakest bond K_B aligned in the same direction. However, this divergence has nothing to do with critical phenomena and is present in any chain with inhomogeneous couplings. The detailed structure of the dispersion relation of the lowest band is due to long-range fluctuations and is characterized by the universal exponent z, $1/\tau \sim k^z$, where $z = 2$.

An alternative way to present this same information is to use the distribution of relaxation times, $\rho(1/\tau)$, which is shown in figure 1(b). The same band structure with widths proportional to τ_0^{-1} is evident. However, it is the detailed structure of $\rho(1/\tau)$ in the lowest band that reveals the universal nature of the relaxation. As $1/\tau \to 0$, it scales as $\rho(1/\tau) \sim (1/\tau)^{d_s/2-1}$, where d_s can be considered as the spectral dimension [10, 11] of the distribution and is related to z as follows

$$d_s = 2d_f/z. \tag{1}$$

Here d_f is the dimension of the system. In translationally invariant models with $d_f = 1$, we have $d_s = d_f$ since $z = 2$ and ρ has an inverse square root divergence as the critical point is approached.

The above models can also be analysed using exact real-space renormalization group techniques [6]. The approach does not rely upon translational invariance and the distribution

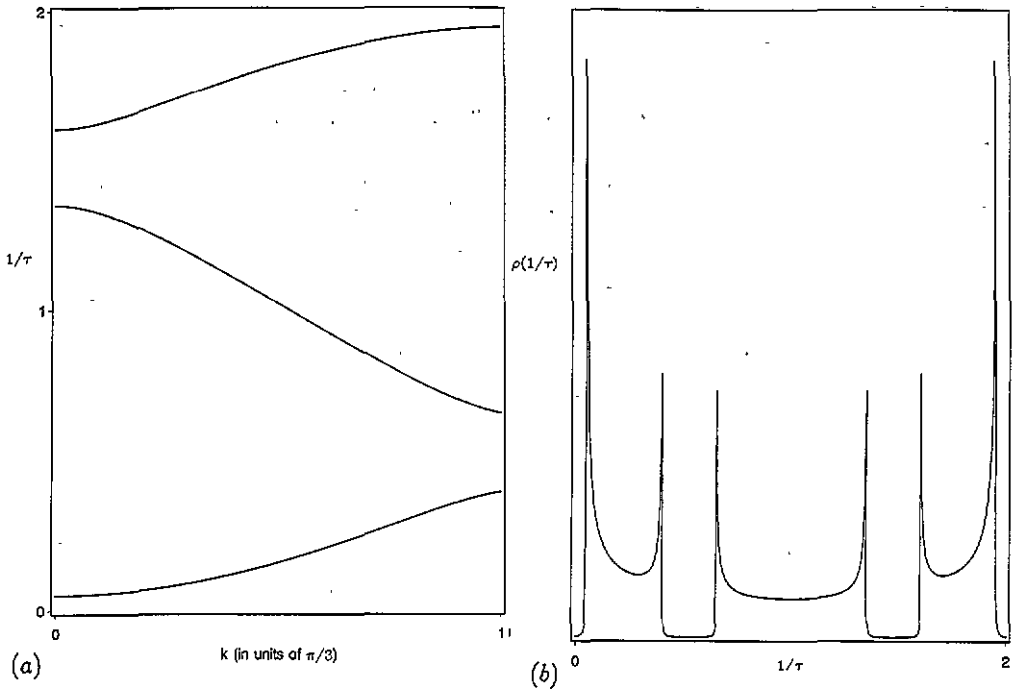

Figure 1. (*a*)The dispersion relation for the inverse relaxation times of a translationally invariant Glauber–Ising chain. The unit cell has three bonds, K_A, K_A, K_B. There are three bands whose width at low temperatures is proportional to $e^{-2|K_B-K_A|}$. (*b*) The corresponding distribution of inverse relaxation times. At the edges of the bands the distribution of relaxation times has an inverse square root divergence.

$\rho(1/\tau)$ can be obtained directly. In this latter approach the short-range time scale appears as a transient of the flow in the parameter space from the initial state. It is the flow away from the unstable fixed point that leads to the universal value $z = 2$.

The study of the critical properties using the distribution of relaxation times is necessary when the system lacks translational invariance. In the present paper we use a generating function method [12] to extend our study of translationally invariant Ising systems with metastable states to systems that are not translationally invariant. Ising models on two different geometries are considered. One is the quasi-periodic Ising chain [13, 14] and the other is the Ising spin system on a 3-simplex gasket [15].

A brief description of the technique used to study $\rho(1/\tau)$ is given in section 2. In sections 3 and 4 the critical dynamics of the quasi-periodic Ising chain and the 3-simplex gasket are studied respectively. Our conclusions are summarized in section 5.

2. The distribution of inverse relaxation times and the RG transformation

The relaxational dynamics of the Ising model is usually described [7] by a master equation for the time-dependent spin probability distribution. This distribution can then be used to study the time evolution of various expectation values of the spins. On the chain the relaxation of the average magnetization is described by a set of linear equations. However, when the coordination number is greater than two, the equations become non-linear in the sense that the expectation values of products of three or more spins are involved. In this

case, we can either try to include these higher-order averages exactly or try to reduce them to linear terms using various approximation schemes. In the models that we consider in this paper, the equations are exactly linear or can be reduced to linear form by identifying the slowest modes. In the following we describe a formalism which can be used to obtain the distribution of modes from such a set of linear equations.

In the particular models that we are studying the variables are the Laplace component $q_l(1/\tau)e^{-t/\tau}$ of the average local magnetization $Q_l(t)$. The equations have the general form

$$\sum_l \left(\left(-\frac{1}{\tau} + E_i \right) \delta_{il} - A_{il} \right) q_l = 0 \tag{2}$$

where $A_{ii} = 0$. The distribution of relaxation times, $\rho(1/\tau)$, can be obtained from the Green function [12], G_{ij}, as follows

$$\rho\left(\frac{1}{\tau}\right) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \operatorname{Im} \frac{1}{N} \sum_i G_{ii} \tag{3}$$

where G_{ij} are the solutions for the inhomogeneous equations,

$$\sum_l \left(\left(-\frac{1}{\tau} - i\epsilon + E_i \right) \delta_{il} - A_{il} \right) G_{lj} = \delta_{lj}. \tag{4}$$

and ϵ is a small imaginary part that has been added to $1/\tau$.

The Green function may be obtained from the complex generating function [12]†

$$\mathcal{F}((1/\tau), A_{ij}, E_i) = \int \mathcal{D}q \exp[i\mathcal{S}^{(0)}((1/\tau), A_{ij}, E_i, q_i)] \tag{5}$$

where

$$\int \mathcal{D}q \equiv \prod_{i=1}^{N} \int_{-\infty}^{\infty} dq_i \qquad \mathcal{S}^{(0)} \equiv \sum_i \frac{1}{2} \left(\frac{1}{\tau} + i\epsilon - E_i^{(0)} \right) q_i^2 + \sum_{ij} A_{ij}^{(0)} q_i q_j \tag{6}$$

and the superscript (0) is introduced for later convenience. From the logarithm of the generating function $\mathcal{F}$ one can obtain the individual G_{ij} by taking partial derivatives with respect to the $A_{ij}^{(0)}$. However, the distribution of relaxation times (3) is obtained directly as follows

$$\rho\left(\frac{1}{\tau}\right) = \frac{1}{N} \frac{2}{\pi} \lim_{\epsilon \to 0^+} \operatorname{Im} \left(\frac{\partial \ln \mathcal{F}(1/\tau + i\epsilon)}{\partial (1/\tau)} \right). \tag{7}$$

The complex generating function plays the same role for the dynamics as the partition function does for the statics. Therefore, the RG techniques that are used to study the static critical behaviour [16] can also be applied to the study of $\rho(1/\tau)$. The simplest kind of RG transformation corresponds to 'decimation' where some fraction of the degrees of freedom are integrated out and new effective interactions between the remaining degrees of freedom are determined. The generating function is then obtained by iterating this procedure until all degrees of freedom have been eliminated or convergence is reached. Using (7) the

† Note that the $\mathcal{F}$ in [12] corresponds to $\ln \mathcal{F}$ in the present article.

distribution of relaxation times can be calculated and plotted as a function of $1/\tau$. However, if we are only interested in the singularities of ρ, then the nature of these are easily obtained from the unstable fixed points or cycles of the RG transformation. It is important [16, 17] to note that *the decimation transformation does not always possess a fixed point even though it can be used to calculate ρ exactly.* In the following we will allow for transformations where the new degrees of freedom are not simply related to the old degrees of freedom by a relabelling. As will be shown explicitly in section 4, such a transformation is necessary to obtain a fixed point for the 3-simplex gasket.

In general our RG transformations consist of three steps. The first step corresponds to multiplying $\mathcal{F}$ by the function $\int \mathcal{D}q^{(1)} \cdot T(q, q^{(1)}) = 1$. This leaves $\mathcal{F}$ invariant and introduces new degrees of freedom $q^{(1)}$. In the case of decimation, this is simply a relabelling of some fraction of the original q's. However, this can also correspond to a decoration process where additional variables are introduced. The ratio between the number of components N in the set $\{q\}$ and the number $N^{(1)}$ in the set $\{q^{(1)}\}$ is

$$N/N^{(1)} \equiv b^{d_f} \tag{8}$$

where b is the scaling factor and d_f is the dimension. The second step consists of interchanging the order of integration and integrating over the original set q to obtain

$$\begin{aligned}\mathcal{F}((1/\tau), A_{ij}, E_i) &= \int \mathcal{D}q^{(1)} \int \mathcal{D}q \cdot T(q, q^{(1)}) \exp[\mathrm{i}\mathcal{S}^{(0)}(1/\tau, A_{ij}, E_i, q_i)] \\ &\equiv \int \mathcal{D}q^{(1)} \exp[\mathrm{i}\mathcal{S}^{(1)}(1/\tau, A_{ij}^{(1)}, E_i^{(1)}, q_i^{(1)})] \exp[NC^{(1)}(1/\tau, A_{ij}^{(0)}, E_i^{(0)})].\end{aligned} \tag{9}$$

The choice of T is made so that the integration over q in (9) can be done exactly. The third step involves a renormalization of the parameters $A_{ij}^{(1)}$ and $E_i^{(1)}$ so that $\mathcal{S}^{(1)}$ has the same functional form as $\mathcal{S}^{(0)}$ and this last step defines the complex constant of integration $C^{(1)}$.

The above transformation can be expressed in the form

$$f(1/\tau, A_{ij}^{(0)}, E_i^{(0)}) = g^{(1)}(1/\tau, A_{ij}^{(0)}, E_i^{(0)}) + b^{-d_f} f(1/\tau, A_{ij}^{(1)}, E_i^{(1)}) \tag{10}$$

where

$$f \equiv \frac{1}{N}\frac{2}{\pi} \lim_{\epsilon \to 0} \mathrm{Im} \ln \mathcal{F}$$

is the integrated density of inverse relaxation times and $g^{(1)} = \mathrm{Im}(2/\pi)C^{(1)}$. A similar relation for ρ can be written using (7), and iterating (10) m times leads to

$$\rho(1/\tau) = \sum_{n=1}^{m-1} b^{-nd_f} \frac{\partial}{\partial(1/\tau)} g^{(n)}(1/\tau, A_{ij}^{(n-1)}, E_i^{(n-1)}) + b^{-md_f} \rho(1/\tau, A_{ij}^{(m)}, E_i^{(m)}). \tag{11}$$

This equation is the basic algorithm for our study of ρ.

In the models that are discussed in the next sections, the trajectory flow of the RG transformation is described by a set of dimensionless parameters $\Gamma^{(n)} = \{\Gamma_i^{(n)}\}$ where the initial values are denoted by $\Gamma^{(0)} = \Gamma^{(0)}(1/\tau, A_{ij}^{(0)}, E_i^{(0)})$. These parameters have the values

Γ^* at the unstable fixed point of the RG transformation and near Γ^* the trajectory flow is described by the linearized RG transformation with the leading eigenvalue λ,

$$\lambda = b^z = b^{2d_f/d_s}. \tag{12}$$

Standard scaling arguments applied to (10) yield the following scaling form for the singular part of ρ

$$\rho(1/\tau) = b^{z-d_f}\rho(b^z/\tau). \tag{13}$$

The fixed point of interest in our case is the one that controls the behaviour of ρ near $1/\tau = 0$. In this case the scaling equation (13) has the solution [4]

$$\rho(1/\tau) \sim (1/\tau)^{d_s/2-1}\mathcal{Q}(\ln 1/\tau). \tag{14}$$

In general, the function $\mathcal{Q}(x)$ is a periodic function with a period $2d_f/d_s \ln b$ and thus can have several non-vanishing Fourier components. In the case of translationally invariant lattices where the scaling factor b can be chosen arbitrarily, it can be argued [4] that $\mathcal{Q}$ is a constant. However, in the case of non-translationally invariant lattices, the scaling factor is not arbitrary and the argument fails. Several groups have used this form to analyse periodic amplitudes for various types of problems [18, 19]. The above solution is only valid within the linear regime of the RG transformation. We shall see in the following sections that some important modifications are required when metastable states are present. The initial values of the parameters in $\mathcal{S}^{(0)}$ are located in the parameter space far away from the unstable fixed point but they flow after the first few RG iterations to the linear regime. This initial flow is characterized by a scale factor, λ_0, which is very different from the λ that characterizes the linear regime and is temperature dependent. This fact does not affect the critical exponent, d_s, appearing in (14) but the amplitude of $\mathcal{Q}(\ln 1/\tau)$ is proportional to λ_0. Using the chain rule of differentiation, ρ can be written as

$$\rho(1/\tau) = \sum_{n=1}^{\infty} b^{-nd_f} \frac{\partial g^{(n)}}{\partial \Gamma^{(n-1)}} \frac{\partial \Gamma^{(n-1)}}{\partial \Gamma^{(n-2)}} \cdots \frac{\partial \Gamma^{(1)}}{\partial \Gamma^{(0)}} \frac{\partial \Gamma^{(0)}}{\partial (1/\tau)} \tag{15}$$

where it is assumed that the last term in (11) does not contribute in the limit $m \to \infty$. In the linear regime of the RG transformation all the derivatives $\partial\Gamma^{(n)}/\partial\Gamma^{(n-1)}$ are approximated by λ. However, for the cases that we consider, the initial derivative $\partial\Gamma^{(1)}/\partial\Gamma^{(0)} \sim \lambda_0$ can diverge at low temperatures. Since $\rho \sim \lambda_0$ and the total number of metastable states is bounded, this means that the width of the lower band should go to zero as $1/\lambda_0$ and hence λ_0 is a new bare time scale in the problem which arises due to the presence of these metastable states.

To emphasize the effect of λ_0, we rewrite equation (14) as

$$\rho(1/\tau) \sim (1/\tau)^{d_s/2-1} A(T) \mathcal{Q}(\ln 1/\tau). \tag{16}$$

The spectral dimension, the period of the 'regular' function $\mathcal{Q}$ and $A(T)$ can all be extracted from the RG transformation. However, each of them describes different features. The spectral dimension, or rather the dynamic exponent (12), characterizes the effect of the long-range fluctuations and is universal. The period of $\mathcal{Q}$ depends on z and the scaling factor b and is hence 'less' universal. The pre-factor $A(T)$ describes non-universal features of the particular model on this geometry. When metastable states are present, this factor is strongly temperature-dependent and can diverge as the critical point is approached. In the next sections we study the Ising model on two quite different geometries where all of these effects are present.

3. The kinetic Ising model on a quasi-periodic chain

The quasi-periodic chain is a geometry which is not translationally invariant, but simple enough to have coordination number 2. Quasi-periodic lattices are defined by the incommensurate length scales of their elementary units, and their self-similarity under certain inflation and deflation rules [13, 14]. One of the most common quasi-periodic chains that has been studied is the Fibonacci tiling which is generated by the following inflation rule from two seeds L and S: each S is replaced by an L, while each L is replaced by an LS. This substitution is iterated until the sequence has the desired length. For instance, a growth sequence over five steps, starting from L, would be

$$L \to LS \to LSL \to LSLLS \to LSLLSLSL. \tag{17}$$

The linear quasi-periodic Ising model [13] is a chain of spins, $\{\sigma_i = \pm 1\}$ which interact with their nearest neighbours on either side. The Hamiltonian (in units of $1/k_B T$) is

$$H = -\sum_i K_i \sigma_{i-1} \sigma_i \tag{18}$$

where the bonds K_i have two values which are distributed in a quasi-periodic fashion along the chain. Denoting these two values as L and S, we assume that the K_i are ordered along the chain according to the Fibonacci ordering (17). An index $\chi = \{\alpha, \beta, \gamma\}$ can be added to each spin site to describe the local environment of the spin: σ^α interacts via two L interactions, σ^β interacts to the right by an S interaction, σ^γ interacts to the left via an S interaction. The relative fractions n_χ of the three kinds of sites is

$$n_\alpha = 2\zeta - 3 \qquad n_\beta = 2 - \zeta \qquad n_\gamma = 2 - \zeta \tag{19}$$

where $\zeta = \frac{1}{2}(1 + \sqrt{5})$ is the golden mean. A more detailed description of the model and a study of its equilibrium properties can be found in Achiam *et al* [13].

The dynamics of the model [14] is described by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\{\sigma\}, t) = -\sum_i (1 - p_i) W_i(\sigma_i) P(\{\sigma\}, t) \tag{20}$$

where $P(\{\sigma\}, t)$ is the time-dependent spin probability distribution. In the above equation p_i is the spin flip operator, $p_i f(\sigma_i) = f(-\sigma_i)$, and the generalized Glauber [7] transition rates are

$$W_i = \tfrac{1}{2}(1 - a_i^+ \sigma_{i-1} \sigma_i - a_i^- \sigma_i \sigma_{i+1}) \tag{21}$$

where

$$a_i^\pm \equiv \tfrac{1}{2}[\tanh(K_i + K_{i+1}) \pm \tanh(K_i - K_{i+1})]. \tag{22}$$

The average value of each spin is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_i \equiv \frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_i \rangle = -2\Gamma \langle \sigma_i W_i(\sigma_i) \rangle \tag{23}$$

where the average $\langle\rangle$ is taken with respect to $P(\{\sigma\},t)$. Using (22) and the Fibonacci ordering, the set of equations (23) for the Laplace component $q_l(1/\tau)e^{-t/\tau}$ of the average local magnetization is reduced to the following three subsets of equations,

$$(-1/\tau + E_\alpha)q_l^\alpha = a_1(q_{l-1} + q_{l+1}) \qquad (-1/\tau + E_\beta)q_l^\beta = a_2 q_{l-1} + a_3 q_{l+1}$$
$$(-1/\tau + E_\gamma)q_l^\gamma = a_4 q_{l-1} + a_5 q_{l+1} \tag{24}$$

where $E_\alpha = E_\beta = E_\gamma = 1$ and

$$a_1 \equiv \tfrac{1}{2}\tanh(2L) \qquad a_2 = a_5 \equiv \tfrac{1}{2}(\tanh(L+S) + \tanh(L-S))$$
$$a_3 = a_4 \equiv \tfrac{1}{2}(\tanh(L+S) - \tanh(L-S)). \tag{25}$$

The set of equations (24) is a set of linear equations of the form (2). A simple RG transformation (9) corresponds to the anti-inflation rule $LS \to L$, $L \to S$ which corresponds to a decimation of the β sites only. The number of remaining sites is ζ^{-1} times the original number and are relabelled according to the following weight function

$$T(q,q') \equiv \prod_{n,m,\ell} \delta(q_n^\alpha - q_n'^\gamma)\delta(q_m^{\gamma(\alpha)} - q_m'^\beta)\delta(q_\ell^{\gamma(\beta)} - q_\ell'^\alpha). \tag{26}$$

The additional index in brackets for the γ sites denotes the type of neighbouring spin to the right. This relabelling allows the new equations to be written in the same form as (24) with renormalized parameters.

The renormalization of the parameters is the following [14]

$$E'_\alpha = E_\gamma - a_4\Gamma_3 - a_5\Gamma_2 \qquad E'_\beta = E_\gamma - a_4\Gamma_3 \qquad E'_\gamma = E_\alpha - a_1\Gamma_2$$
$$a'_1 = a_4\Gamma_2 \qquad a'_2 = a_4\Gamma_2 \qquad a'_3 = a_5 \qquad a'_4 = a_1 \qquad a'_5 = a_1\Gamma_3 \tag{27}$$

where

$$\Gamma_1 \equiv \frac{a_1}{-1/\tau - i\epsilon + E_\alpha} \qquad \Gamma_2 \equiv \frac{a_2}{-1/\tau - i\epsilon + E_\beta} \qquad \Gamma_3 \equiv \frac{a_3}{-1/\tau - i\epsilon + E_\beta}$$
$$\Gamma_4 \equiv \frac{a_4}{-1/\tau - i\epsilon + E_\gamma} \qquad \Gamma_5 \equiv \frac{a_5}{-1/\tau - i\epsilon + E_\gamma}. \tag{28}$$

The recursion relations for these dimensionless parameters are

$$\Gamma'_1 = \frac{\Gamma_2\Gamma_4}{1 - \Gamma_3\Gamma_4 - \Gamma_2\Gamma_5} \qquad \Gamma'_2 = \frac{\Gamma_2\Gamma_4}{1 - \Gamma_3\Gamma_4} \qquad \Gamma'_3 = \frac{\Gamma_5}{1 - \Gamma_3\Gamma_4}$$
$$\Gamma'_4 = \frac{\Gamma_1}{1 - \Gamma_1\Gamma_2} \qquad \Gamma'_5 = \frac{\Gamma_1\Gamma_3}{1 - \Gamma_1\Gamma_2}. \tag{29}$$

Under the decimation transformation, the $\{a_i\} \to 0$, and we can obtain the G_{jj} (*of a particular site* j) [12] from

$$G_{jj} = \lim_{n\to\infty} \frac{1}{-1/\tau - i\epsilon + E_j^{(n)}}. \tag{30}$$

However, the sites that remain after many iterations are only a small subset of the original sites. Because the lattice is not translationally invariant, these sites alone cannot be used to obtain the total distribution of relaxation times (3). The imaginary parts of these G_{jj} will give information about the nature of the distribution including the singularities and the locations of gaps. However, there is no guarantee that all modes will be present in these response functions. In systems without translational invariance, all of the G_{ii} should be calculated to obtain the total distribution [20]. Ashraff and Stinchcombe [19] used these three local G_{jj} to study the related problem of electrons and phonons on quasi-periodic chains. The local response functions were weighted as follows

$$\rho(1/\tau) = (1/\pi)\,\mathrm{Im}(n_\alpha G_{\alpha\alpha} + n_\beta G_{\beta\beta} + n_\gamma G_{\gamma\gamma}). \tag{31}$$

Since all of the sites in each subgroup (α, β, γ) are not equivalent, this represents an approximation to the total distribution. For this reason, we must use the generating function approach described in section 2.

Equations (24) can be expressed entirely in terms of the Γ_i by making the coefficient of q^χ on the left-hand side equal to unity. Similarly, in the expression for $S^{(0)}$ we take the coefficient of q_i^2 to be equal to $-\frac{1}{2}$ at each step of the transformation. This ensures that $S^{(0)}$ always has the same functional form and identifies an initial constant $C^{(0)}$ as

$$C^{(0)} = -\tfrac{1}{2}\ln[d_\alpha^{n_\alpha} d_\beta^{n_\beta} d_\gamma^{n_\gamma}] \tag{32}$$

where $d_\chi = 1/\tau + \mathrm{i}\epsilon - E_\chi^{(0)}$. The summation in (15) is extended to include this initial term which represents the distribution for non-interacting spins. At subsequent steps $C^{(n)}$ is given by

$$C^{(n+1)} = -\tfrac{1}{2}\ln[(1 - \Gamma_3^{(n)}\Gamma_4^{(n)} - \Gamma_2^{(n)} - \Gamma_5^{(n)})^{n_\alpha}(1 - \Gamma_3^{(n)}\Gamma_4^{(n)})^{n_\beta}(1 - \Gamma_1^{(n)}\Gamma_2^{(n)})^{n_\gamma}]. \tag{33}$$

The RG iteration procedure (27)–(29) and expression (15) provide an efficient numerical procedure for calculating ρ. The resulting distribution of relaxation times is plotted in figure 2 for $L = 1.0$ and $S = 0.75$. There are three main self-similar bands separated by large gaps. In this respect, the spectrum is similar to that of the periodic chain shown in figure 1(b). However, for the quasi-periodic chain, this splitting of the spectrum occurs repeatedly within each of these main bands. At $T = 0$, this model has metastable states which correspond to the spins separated by the weaker of the two bonds all pointing in the same, but arbitrary, direction. At low temperatures, these three main bands become very narrow with the lowest band describing the metastable states. Figure 3(a) shows a ln–ln plot of the distribution for the same ratio of S to L as in figure 2. but with $L = 5.0$. The behaviour is in accord with (16) and the slope yields $\frac{1}{2}d_s - 1 = -\frac{1}{2}$ which corresponds to $z = 2$. The only significant Fourier component of the function $\mathcal{Q}$ as $1/\tau \to 0$ is the constant term. The structure at larger values of $1/\tau$ is due to the singularities at the gap edges and the scaling behaviour occurs about finite values of $1/\tau$.

Below we will show explicity that the non-universal amplitude $A(T)$ in (16) is given by $A(T) = \mathrm{e}^{2(L-S)}$. In figure 3($b$) we illustrate this by plotting $\ln \Lambda\rho$ against $\ln 1/\Lambda\tau$ where $\Lambda = A(T_1)/A(T_2)$. The temperature T_1 is the same as in figure 3(a) and $T_2 = 0.5T_1$. Since the two plots in figures 3(a) and (b) coincide, this confirms the assertion in section 2 that the amplitude in (16) is non-universal and that it introduces a new bare time scale for the inverse relaxation time $1/\tau$. However the critical exponent z remains universal.

Ashraff and Stinchcombe [14] have studied the RG transformation (29) in detail and concluded that z was non-universal. The transformation has a critical fixed point

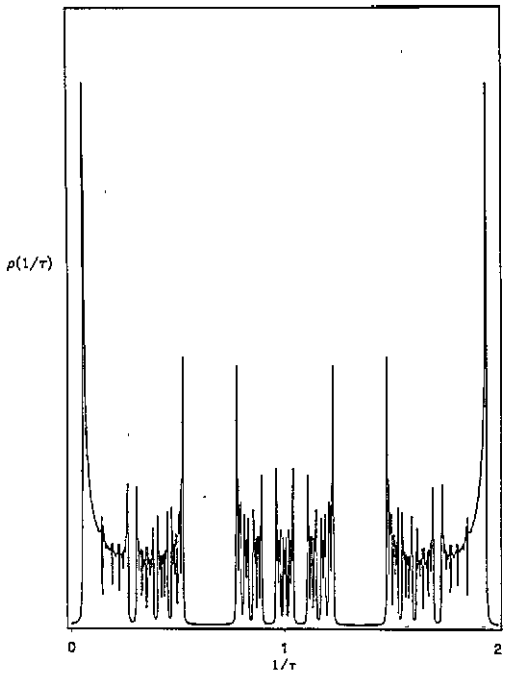

Figure 2. The distribution of inverse relaxation times of the quasi-periodic Glauber–Ising chain for $L = 1.0$, $S = 0.75$ and $\epsilon = 10^{-3}$. There are three main bands with an internal self-similar structure.

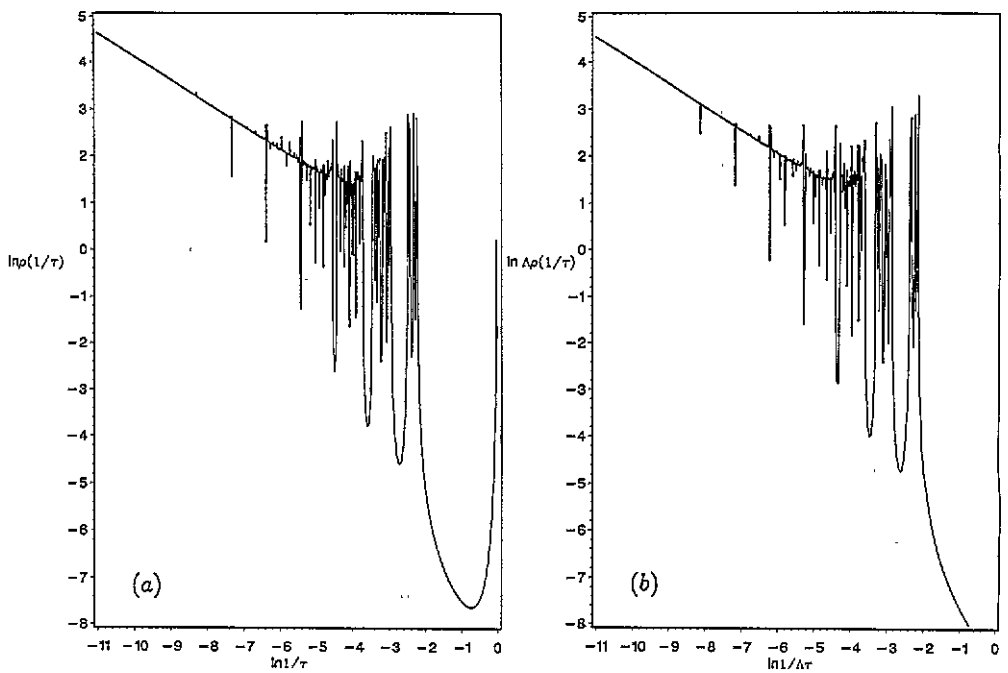

Figure 3. (*a*) A ln–ln plot of the distribution of relaxation times of the quasi-periodic Glauber–Ising chain. The slope indicates a $(1/\tau)^{-1/2}$ divergence of ρ. (*b*) The same plot at a lower temperature than in (*a*) with $T_2 = 0.5T_1$. Both ρ and τ have been scaled by $\Lambda = A(T_1)/A(T_2)$.

$\Gamma^* = (\frac{1}{2}, \zeta^{-2}, \zeta^{-1}, \zeta^{-1}, \zeta^{-2})$ which describes the behaviour of the system near $1/\tau = 0$.

The leading eigenvalue of the linear RG transformation is ζ^2, and this yields the universal value $z = 2$ as in the case of all $d = 1$ Ising models. Therefore, according to (16) ρ diverges as $\rho \sim (1/\tau)^{-1/2}$. This universal behaviour is confirmed in figure 3(a). The metastable states in this model introduce a new 'bare' time scale which should be separated from the effects of the long-range fluctuations. At low temperatures and near $1/\tau = 0$, the initial value of $\Gamma^{(0)} \sim \{\frac{1}{2}, 1-\theta, \theta, \theta, 1-\theta\}$ where $\theta = e^{-2(L-S)} \ll 1$ (we assume $L > S$). At the first iteration, Γ_2 and Γ_3 interchange their values but Γ_1 stays fixed at $\frac{1}{2}$ while the relationships $\Gamma_2 = \Gamma_5$, $\Gamma_3 = \Gamma_4$ are maintained. At the second iteration, all of the Γ_i are in the vicinity of the fixed point. In this region the flow can be described as a run away from the fixed point with the temperature-independent scale factor ζ^2. The first two iterations result in both $\partial\Gamma^{(1)}/\partial\Gamma^{(0)}$ and $\partial\Gamma^{(2)}/\partial\Gamma^{(0)}$ being proportional to $1/\theta$. According to the discussion in section 2, the amplitude $A(T)$ is proportional to $e^{2(L-S)}$. The width of the band is inversely proportional to this initial scale factor, and thus has a non-universal temperature-dependent value.

Two initial iterations are required to scale all the parameters to the linear regime which is due to an interchange of the two bonds when the sites are relabelled at each step. Similar behaviour is also present in the static problem [13]. It is important that only two iterations are needed for the Γ to reach the linear region. This fact clearly separates λ_0 from the repulsive eigenvalue of the fixed point. Ashraff and Stinchcombe [14] did not separate these factors and hence they concluded that z was non-universal.

The regular amplitude $Q(x)$ in (16) may be expected to have an oscillatory behaviour with a period $z \ln \zeta \sim 0.96$. The appearance of a constant $Q(x)$ as $1/\tau \to 0$ (figure 3) is probably due to the fact that at low temperatures the configurations with the largest τ are those in which the spins between the weak bonds are frozen into a parallel alignment. Hence the dynamics of the chain is essentially equivalent to a uniform-bond Ising chain where only the weaker bond provides the coupling between blocks of spins with a new 'bare' flipping rate. Since the uniform chain can be rescaled by any scaling factor, oscillations in the amplitude would not appear [4].

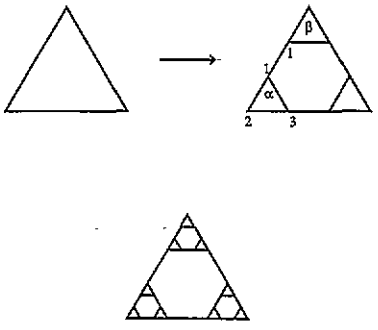

Figure 4. The triangular generator of the 3-simplex lattice and the fractal after two iterations. The sites within each elementary triangle (α) are labelled by 1, 2, 3.

4. The kinetic Ising model on a 3-simplex lattice

The 3-simplex lattice is a fractal ($d_f = \ln 3/\ln 2$) with the triangular generator as shown in figure 4. The spins $\{\sigma_i^\alpha = \pm 1\}$ are located at the nodes of the 3-simplex lattice. The index α denotes the 'cell' (elementary triangle) and $i = \{1, 2, 3\}$ denotes the corners of that triangle. The spins interact according to the Hamiltonian (18) with equal bonds, $K_i = K$,

and the dynamics of the model is described by the generalized Glauber dynamics which obeys the master equation (20) with the transition rate

$$W_i^\alpha = \tfrac{1}{2}(1 - \sigma_i^\alpha[a(\sigma_{i+1}^\alpha + \sigma_{i+2}^\alpha + \sigma_i^\beta) + b\sigma_{i+1}^\alpha\sigma_{i+2}^\alpha\sigma_i^\beta]) \tag{34}$$

where

$$a = \tfrac{1}{4}(\tanh 3K + \tanh K) \qquad b = \tfrac{1}{4}(\tanh 3K - 3\tanh K). \tag{35}$$

The nearest-neighbour spins σ_i^α and σ_i^β are located in two different cells and the index i of σ_i^α is defined modulus 3.

Since the coordination number of the system is 3, the kinetic equations (23) for the average local magnetization include the non-linear term $\langle\sigma_{i+1}^\alpha\sigma_{i+2}^\alpha\sigma_i^\beta\rangle$. To approximate this term we focus on the metastable states that control the slowing down near $T = 0$. In the 3-simplex geometry the metastable states are formed by the configurations with the three spins in each cell pointing in the same direction [15, 21]. Therefore at low temperatures we have $\sigma_{i+1}^\alpha\sigma_{i+2}^\alpha\sigma_i^\beta \approx \sigma_i^\beta$. Using this relation, the non-linear term is replaced by $\langle\sigma_i^\beta\rangle$ and the kinetic equations for the spins in the α triangle become

$$\begin{aligned}
&(-1/\tau + E)q_1^\alpha = a_1(q_2^\alpha + q_3^\alpha) + a_2 q_1^\beta \qquad (-1/\tau + E)q_2^\alpha = a_1(q_3^\alpha + q_1^\alpha) + a_2 q_2^\gamma \\
&(-1/\tau + E)q_3^\alpha = a_1(q_2^\alpha + q_1^\alpha) + a_2 q_3^\delta
\end{aligned} \tag{36}$$

where $a_1 = a, a_2 = a + b$ and $E = 1$. Note that $a_2 \to 0$ as $T \to 0$ and the triangles completely decouple.

The exponent $\mathcal{S}^{(0)}$ in the generating function can be divided into intra- and inter-cell terms as follows

$$\mathcal{S}^{(0)} = \sum_\alpha[-\tfrac{1}{2}(q_1^{\alpha 2} + q_2^{\alpha 2} + q_3^{\alpha 2}) + \Gamma_1^{(0)}(q_1^\alpha q_2^\alpha + q_2^\alpha q_3^\alpha + q_1^\alpha q_3^\alpha)] + \Gamma_2^{(0)}\sum_{\langle\alpha,\beta\rangle} q_i^\alpha q_i^\beta \tag{37}$$

where now $\Gamma_1^{(0)} = a_1/(1 - 1/\tau - i\epsilon)$ and $\Gamma_2^{(0)} = a_2/(1 - 1/\tau - i\epsilon)$. As in the previous section, we define the Γ at each stage by requiring the coefficient of q_i^2 to be $-\frac{1}{2}$. This defines an initial term in the expression (15) for ρ given by $C^{(0)} = -\frac{1}{2}\ln(1/\tau + i\epsilon - 1)$ which represents the non-interacting contribution.

One possible RG approach to calculate ρ is to use a decimation or dedecoration transformation [16], where the spins on the smallest triangles are eliminated. However, for this lattice the decimation approach does not have a fixed point and the separation of non-universal amplitudes is difficult. This difficulty also occurs for other types of problem on both this lattice and the linear chain [17]. For this reason we will use a block spin, or decoration, transformation which corresponds to introducing a new spin variable q_α for each of the elementary triangles. The weight function has the form $T = \prod_{\alpha=1}^{N/3} T^\alpha$ where

$$T^\alpha(q^\alpha, q_\alpha) = \frac{1}{\sqrt{2\pi i}}\exp\{-\tfrac{1}{2}i[q_\alpha - \Gamma_1^{(0)1/2}(q_1^\alpha + q_2^\alpha + q_3^\alpha)]^2\}. \tag{38}$$

All of the original sites on the lattice can be eliminated exactly leaving a lattice of block spins with the same structure as the original lattice corresponding to a scale factor $b = 2$. At the first iteration Γ_1 and Γ_2 become equal

$$\Gamma_1^{(1)} = \Gamma_2^{(1)} = \frac{\Gamma_1^{(0)}\Gamma_2^{(0)}}{1 - \Gamma_1^{(0)} - 2\Gamma_1^{(0)2} - \Gamma_2^{(0)2}} \tag{39}$$

Further transformations can be described by the one-parameter map

$$\Gamma^{(n+1)} = \frac{\Gamma^{(n)2}}{1 - \Gamma^{(n)} - 3\Gamma^{(n)2}}. \tag{40}$$

The critical fixed point of (40) is $\Gamma^* = \{\frac{1}{3}, \frac{1}{3}\}$ and the linearized RG has a repulsive eigenvalue $\lambda = 5$. Using (12) the critical dynamic exponent is found to have the value $z = \ln 5/\ln 2$. The distribution of relaxation times is calculated using (15) and the following expressions for $C^{(n)}$

$$C^{(1)} = -\tfrac{1}{12}\ln[(1+\Gamma_1^{(0)}-\Gamma_2^{(0)})(1+\Gamma_1^{(0)}+\Gamma_2^{(0)})] - \tfrac{1}{6}\ln(1-\Gamma_1^{(0)}-2\Gamma_1^{(0)2}-\Gamma_2^{(0)2})$$

$$C^{(n)} = -\tfrac{1}{12}\ln(1+2\Gamma^{(n)}) - \tfrac{1}{6}\ln(1-\Gamma^{(n)}-3\Gamma^{(n)2}) \tag{41}$$

and is plotted in figure 5 for $K = 1.0$. In the limit of low temperatures, the two 'bands' shrink in width with the lower band describing the slowly relaxing metastable states. As $1/\tau \to 0$ the distribution of relaxation times diverges as in (16) with $d_s = 2\ln 3/\ln 5$. This same value of d_s is the spectral dimension for electronic and scalar vibrational problems on the 3-simplex or Sierpinski gasket lattices [10, 11, 17, 22].

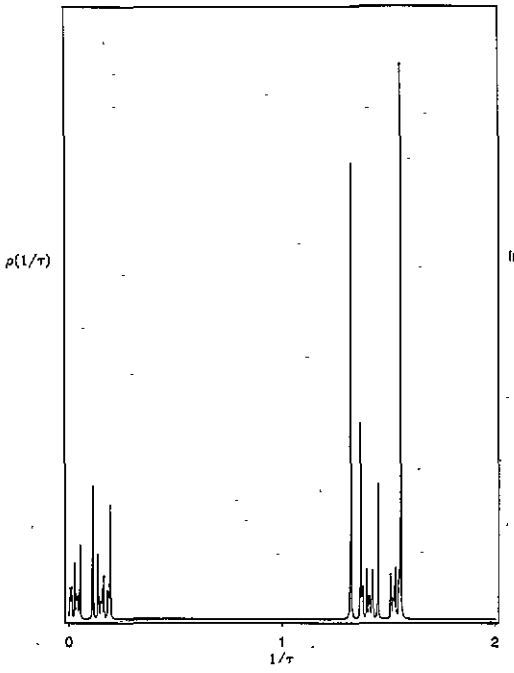

Figure 5. The distribution of inverse relaxation times of the 3-simplex Glauber–Ising model for $K = 1.0$ and $\epsilon = 10^{-3}$. There are two main 'bands' which are self-similar.

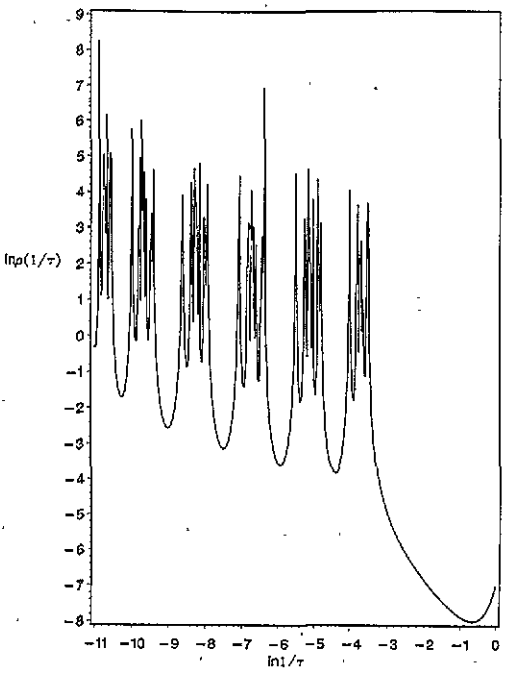

Figure 6. A ln–ln plot of the distribution of relaxation times of the 3-simplex lattice at a lower temperature than in figure 5 and corresponding to $K = 2.0$. The slope agrees with the value $(\ln 3/\ln 5 - 1)$ obtained from the fixed point. The function Q has many Fourier components with period $\ln 5$.

The non-universal amplitude $A(T)$ is obtained in the same way as it was for the quasi-periodic chain. The initial values of the Γ are $\{(1-\theta)/2, \theta\}$ where $\theta = e^{-2K}$. In the first step of the RG transformation Γ moves into the vicinity of the unstable fixed point $\{\frac{1}{3}, \frac{1}{3}\}$. The partial derivative λ_0 discussed in section 2 is proportional to θ^{-1} and hence diverges as $T \to 0$. Figure 6 shows a ln–ln plot of the distribution at a lower temperature corresponding to $K = 2.0$. The oscillations occur about a slope which agrees with the value of $\frac{1}{2}d_s - 1$ obtained from the fixed point and is universal. The function $\mathcal{Q}$ is periodic with period $z \ln b = \ln 5$ and has many Fourier components. The period is also universal but depends explicitly on the fact that the geometry is only self-similar under scaling by $b = 2$.

5. Conclusion

We have found that the distribution of inverse relaxation times is a useful tool for studying critical dynamics in systems which are not translationally invariant. The distribution can be obtained numerically from a generating function using RG methods and the behaviour of the distribution as $1/\tau \to 0$ yields the dynamic critical exponent z. If the system has metastable states then the critical amplitude contains a non-universal contribution which must be extracted since it defines a new 'bare' time scale for the dynamics at low temperatures. The dynamic exponent and the divergence of the distribution of relaxation times are related to the spectral dimension of the system which also characterizes the density of states of a variety of other problems. These include models for electrons, scalar elasticity and electrical conduction on the quasi-periodic chain [19, 22–24] and Sierpinski gasket [22, ,10, 11, 25].

An important feature in the models that we studied is that the width of the lowest band tends to zero as the temperature approaches zero. This narrowing of the band gives an additional contribution to the critical slowing down since a new time scale is introduced. In conventional numerical simulations of the scaling properties of these models it will be hard to distinguish between the two mechanisms that affect the relaxation towards equilibrium. If they are not properly separated, the numerical experiments would find a non-universal value of z [26, 27].

We have related three separate features of the distribution of relaxation times, to the behaviour of the RG transformation. The universal exponent, z, is related to the leading eigenvalue of the critical fixed point of the transformation. The oscillations of the amplitude have a period that is related to z and to the geometrical properties of the lattice. Finally, the diverging bare time scale can be obtained from the transients of the non-linear RG transformation. This identification provides a possible solution to the problems encountered by Kutasov *et al* [27] in their study of a model with hierachical couplings.

The results for the critical slowing down obtained in this study are different from the model proposed by Henley [8] for critical percolation clusters. In our case, the non-universal 'bare' time scale does not depend on the system size. This behaviour is due to the fact that the metastable states in these systems are compact groups of spins separated by the weak bonds. In the case of the fractal lattices considered by Kutasov *et al* [26], we have not been able to identify similar compact groups of spins. For these lattices, it is possible that the non-universal factor λ_0 contributes at all steps of the RG transformation and this would indeed be consistent with the model proposed by Henley.

Acknowledgment

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References

[1] See, e.g. 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic) and references therein

[2] Halperin B I and Hohenberg P C 1969 *Phys. Rev.* **177** 952
Ferrell R A, Menyhárd N, Schmidt H, Schwabl F and Szépfalusy P 1968 *Ann. Phys., NY* **47** 565

[3] Halperin B I, Hohenberg P C and Ma S K 1972 *Phys. Rev. Lett.* **29** 1548; 1974 *Phys. Rev.* B **10** 139
Hohenberg P C and Halperin B I 1977 *Rev. Mod. Phys.* **49** 435

[4] Niemeijer T and van Leeuwen J M J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic) p 425

[5] Droz M, Kamphorst Leal da Silva J, Malaspinas A and Stella A L 1987 *J. Phys. A: Math. Gen.* **20** L387

[6] Southern B W and Achiam Y 1993 *J. Phys. A: Math. Gen.* **26** 2505

[7] Glauber R J 1963 *J. Math. Phys.* **4** 294

[8] Henley C L 1985 *Phys. Rev. Lett.* **54** 2030; 1986 *Phys. Rev.* B **33** 7675

[9] Achiam Y and Southern B W 1992 *J. Phys. A: Math. Gen.* **25** L769

[10] Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625

[11] Rammal R and Toulouse G 1983 *J. Physique Lett.* **44** L13

[12] Langlois J-M, Tremblay A-M S and Southern B W 1983 *Phys. Rev.* B **28** 218
Tremblay A-M S and Southern B W 1983 *J. Physique Lett.* **44** L843

[13] Achiam Y, Lubensky T C and Marshall E W 1986 *Phys. Rev.* B **33** 6460

[14] Ashraff J A and Stinchcombe R B 1989 *Phys. Rev.* B **40** 2278

[15] Bell S C and Southern B W 1988 *Phys. Rev.* B **38** 333

[16] Nelson D R and Fisher M E 1975 *Ann. Phys.* **91** 226

[17] Knežević M and Southern B W 1986 *Phys. Rev.* B **34** 4966
Southern B W and Knežević M 1987 *Phys. Rev.* B **35** 5036

[18] Bessis D, Geronimo J S and Moussa P 1983 *J. Physique Lett.* **44** L977

[19] Ashraff J A and Stinchcombe R B 1988 *Phys. Rev.* B **37** 5723

[20] Andrade R F S and Schellnhuber H J 1991 *Phys. Rev.* B **44** 13 213

[21] Cherepanov V B, Cyr S L M and Southern B W 1992 *J. Phys. A: Math. Gen.* **25** 4347

[22] Domany E, Alexander S, Bensimon D and Kadanoff L P 1983 *Phys. Rev.* B **28** 3110

[23] Kohmoto M and Banavar J R 1986 *Phys. Rev.* B **34** 563

[24] Lu J P, Odagaki T and Birman J L 1986 *Phys. Rev.* B **33** 4809
Mori F and Rodriquez J P 1986 *Phys. Rev.* B **34** 2207

[25] Gefen Y, Aharony A, Shapir Y and Mandelbrot B B 1984 *J. Phys. A: Math. Gen.* **17** 435

[26] Kutasov D, Domany E and Pytte E 1987 *Phys. Rev.* B **35** 3354

[27] Kutasov D, Aharony A, Domany E and Kinzel W 1986 *Phys. Rev. Lett.* **56** 2229 and references therein

[28] Achiam Y 1985 *Phys. Rev.* B **31** 4732